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Fuzzy Logic Modelling for Predictive Analysis in Nonlinear Dynamical Systems

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***Abstract**:* Nonlinear dynamical systems, including chaotic systems that exhibit sensitive dependence on initial conditions, present a persistent challenge for predictive modelling because their global input–output behaviour cannot be adequately captured by a single linear model. This paper develops and evaluates a Takagi–Sugeno (T–S) fuzzy modelling framework for one-step-ahead and multi-step predictive analysis of nonlinear dynamical systems, in which the state space is partitioned by overlapping fuzzy sets and a locally linear model is identified within each fuzzy region using weighted least-squares regression, with the overall prediction formed as a fuzzy-weighted blend of the local models. The framework is demonstrated on the logistic map, a canonical low-dimensional chaotic dynamical system, operating in its chaotic regime ($r = 3.9$). Using a three-rule T–S fuzzy model (Low, Medium, High fuzzy partitions of the state variable) identified from 220 training samples and evaluated on 99 out-of-sample test points, the fuzzy model achieves a one-step-ahead test RMSE of 0.088, compared with 0.258 for a single global linear (crisp autoregressive) baseline fitted on the same data — a reduction of roughly 66%. Multi-step iterated prediction is also examined and shown to diverge from the true trajectory within approximately five to seven steps, consistent with the fundamental predictability limits imposed by the positive Lyapunov exponent of the chaotic system rather than any deficiency of the fuzzy model itself. The results demonstrate that fuzzy logic modelling provides a substantial and quantifiable improvement over linear predictive models for nonlinear dynamical systems in the short-to-medium prediction horizon, while also making explicit the point beyond which chaotic dynamics render further improvement in prediction accuracy fundamentally, rather than merely practically, limited.

Keywords: *Fuzzy logic modelling; Takagi–Sugeno fuzzy systems; Nonlinear dynamical systems; Chaos prediction; Logistic map; Predictive analysis; Time series forecasting*.

1. Introduction: Many physical, biological, and engineering processes are governed by nonlinear dynamics, ranging from population growth and epidemiological spread to electrical circuits, mechanical vibration, and climate systems. A defining feature of a broad and important subclass of these systems is deterministic chaos: bounded, aperiodic behaviour generated by entirely deterministic rules, which nonetheless exhibits extreme sensitivity to initial conditions and is therefore only predictable over a finite horizon. Classical linear predictive models — autoregressive models, linear state-space models, and related techniques — are mathematically convenient but structurally unable to represent the curvature and regime-dependent behaviour that characterises nonlinear and chaotic dynamics, typically yielding large and systematic prediction errors when applied outside a narrow operating range.

Fuzzy logic modelling offers an alternative that sits between fully linear and fully nonlinear black-box approaches. The Takagi–Sugeno (T–S) fuzzy model, introduced by Takagi and Sugeno (1985) and further developed by Sugeno and Kang (1988), represents a nonlinear system as a set of fuzzy IF–THEN rules, each associating a fuzzy region of the input space with a locally valid linear (or affine) consequent model. The overall model output is formed by smoothly blending the outputs of the locally linear rules according to their membership degrees, which allows the composite model to approximate globally nonlinear behaviour while retaining the interpretability and computational simplicity of linear regression within each local region. This paper applies the T–S fuzzy modelling paradigm specifically to the problem of predictive analysis in nonlinear dynamical systems, using the logistic map operating in its chaotic regime as a canonical, mathematically well-understood test case. The logistic map, popularised by May (1976) as a simple demonstration of how deterministic nonlinear iteration can generate complex, chaotic dynamics, is a standard benchmark in the nonlinear time-series prediction literature precisely because its true generating equation is known, which allows prediction error to be attributed unambiguously to modelling limitations rather than to unmodelled exogenous noise.

The contributions of this paper are: (1) a complete T–S fuzzy modelling framework for one-step-ahead and iterated multi-step prediction of nonlinear dynamical systems, including explicit weighted least-squares identification of local rule consequents; (2) a fully worked numerical case study on the chaotic logistic map, including a direct, quantified comparison against a linear autoregressive baseline; and (3) an explicit analysis of the multi-step prediction horizon, distinguishing genuine modelling error from the fundamental predictability limits imposed by chaotic sensitivity to initial conditions. The paper is organised as follows. Section 2 reviews relevant literature on fuzzy modelling and chaotic dynamical systems. Section 3 presents preliminaries on nonlinear dynamics and Takagi–Sugeno fuzzy models. Section 4 describes the proposed modelling and identification framework. Section 5 presents the logistic map case study and numerical results. Section 6 discusses the findings, and Section 7 concludes.

2. Literature Review: Zadeh (1965) established the mathematical foundation of fuzzy sets, and Takagi and Sugeno (1985) subsequently proposed representing nonlinear systems through a rule base in which each rule's antecedent defines a fuzzy operating region and each rule's consequent is a locally linear function of the inputs, an architecture now generally referred to as the Takagi–Sugeno (T–S) fuzzy model. Sugeno and Kang (1988) extended this structure-identification problem, proposing systematic procedures for determining both the fuzzy antecedent partitions and the linear consequent parameters directly from input–output data, which is the general approach adopted in the present paper's identification stage.

Wang and Mendel (1992) proposed a related data-driven rule-generation method for extracting fuzzy IF–THEN rules directly from numerical examples, contributing to a broader movement toward automated, data-driven fuzzy model identification rather than purely expert-elicited rule bases. Jang (1993) introduced the adaptive-network-based fuzzy inference system (ANFIS), which embeds a T–S-type fuzzy model within a neural-network training architecture, allowing both antecedent membership function parameters and consequent linear coefficients to be tuned jointly through gradient-based learning; ANFIS and its variants remain among the most widely used fuzzy modelling tools for nonlinear system identification and prediction. Babuška (1998) provided a comprehensive treatment of fuzzy modelling for control and identification, formalising much of the methodology — including fuzzy clustering-based partitioning and local linear regression — that underlies contemporary T–S modelling practice.

On the dynamical-systems side, May (1976) demonstrated that the logistic map, despite being generated by an extremely simple one-dimensional quadratic recursion, exhibits period-doubling bifurcations and

fully developed chaos as its growth-rate parameter increases, making it one of the most widely used low-dimensional benchmarks for nonlinear time-series analysis. Lorenz (1963) had earlier shown, in a three-dimensional continuous system derived from atmospheric convection equations, that deterministic nonlinear systems could generate aperiodic, sensitively-dependent trajectories now recognised as the hallmark of chaos; the fundamental limit this places on long-horizon prediction, regardless of model class, is directly relevant to the multi-step prediction analysis undertaken in Section 5 of this paper.
A substantial body of applied work has combined fuzzy or neuro-fuzzy modelling with chaotic and nonlinear time-series prediction, generally reporting that fuzzy and neuro-fuzzy models outperform linear autoregressive baselines on short-horizon prediction accuracy while nonetheless remaining subject to the same long-horizon predictability limits as any other model class once trajectories diverge beyond the system's characteristic Lyapunov time. The present paper contributes to this literature a fully transparent, reproducible worked example — with explicit rule parameters, weighted least-squares identification, and a direct linear-baseline comparison — intended to make the mechanics and the limits of fuzzy predictive modelling of chaotic systems concrete rather than only reporting aggregate accuracy figures.

3. Preliminaries

3.1 Nonlinear Dynamical Systems and the Logistic Map

A discrete-time nonlinear dynamical system is defined by an iterative map x_{n+1} = f (x_n), where f is a nonlinear function of the state x_n. The logistic map, used as the case-study system in this paper, is defined by:

$$x_{n+1} = r \cdot x_n \cdot (1 - x_n),\ x_n, \in [0, 1]$$

where r is a growth-rate parameter. For r values beyond approximately 3.57, the logistic map enters a chaotic regime characterised by aperiodic trajectories, a positive Lyapunov exponent, and sensitive dependence on initial conditions, meaning that two trajectories starting arbitrarily close together diverge exponentially over time. This paper uses r = 3.9, a value well within the chaotic regime, as the target system for predictive modelling.

3.2 Takagi–Sugeno Fuzzy Model

A first-order Takagi–Sugeno fuzzy model consists of a set of M rules of the form:

$$\textit{Rule } i\text{: IF } x \text{ is } A_i \text{ THEN } y_i = a_i \cdot x + b_i,\quad i = 1, \ldots, M$$

where A_i is a fuzzy set defining the antecedent (input) region of rule i, and a_i, b_i are the parameters of the locally linear consequent model. Given an input x, the firing strength (membership degree) w_i(x) = μ_{A_i}(x) of each rule is computed, and the overall model output is the weighted average of the individual rule outputs:

$$\hat{y}(x) = \Sigma_i\, w_{i(X)} \cdot y_{i(X)} / \Sigma_i\, w_{i(X)} = \Sigma_i\, w_{i(X)} \cdot (a_i\, x + b_i) / \Sigma_i\, w_{i(X)}$$

This structure allows the model to behave as a smooth interpolation between locally linear approximations, each valid within its own fuzzy operating region, thereby approximating globally nonlinear behaviour without requiring an explicit nonlinear functional form to be specified in advance.

3.3 Weighted Least-Squares Identification of Consequent Parameters

Given a training data set $\{(x_k, y_k\}$, k = 1, …, N, the consequent parameters (a_i, b_i) of rule i are identified by weighted least-squares regression, minimising the membership-weighted sum of squared errors:

$$(a_i, b_i) = argmin\_\{a,b\}\ \Sigma\_k\ w\_i(x\_k) \cdot [y\_k - (a \cdot x\_k + b)]^2$$

which admits a closed-form solution analogous to ordinary weighted linear regression. This local, per-rule identification procedure is computationally inexpensive and, unlike gradient-based joint training (as used in ANFIS), does not require iterative optimisation, making it well suited to the transparent, reproducible case study developed in Sections 4 and 5.

Table 1. Fuzzy Partition of the State Variable

Fuzzy Set	Linguistic Meaning	Triangular Parameters (a, b, c)
Low	State value near 0	(0, 0, 0.5)
Medium	State value near 0.5	(0, 0.5, 1)
High	State value near 1	(0.5, 1, 1)

4. Research Methodology

The proposed framework identifies and evaluates a T–S fuzzy predictive model for a nonlinear dynamical system in five stages.

Step 1 — Data generation and partitioning.

A time series $\{x_0, x_1, \ldots, x_n\}$ is generated from the target nonlinear system (or observed from a real process), after discarding an initial transient to ensure the series lies on the system's attractor. Input–output training pairs (x_n, x_{n+1}) are formed and split, in time order, into a training set used for model identification and a held-out test set used for out-of-sample evaluation, preserving the temporal ordering to avoid information leakage from future to past.

Step 2 — Fuzzy partitioning of the state space.

The state variable domain is partitioned into M overlapping triangular fuzzy sets (Table 1 shows the three-rule partition used in the case study: Low, Medium, High), chosen so that at any point in the domain at least one, and generally two, fuzzy sets have non-zero membership, ensuring smooth transitions between locally linear regions.

Step 3 — Local model identification.

For each fuzzy rule i, the locally linear consequent parameters (a_i, b_i) are identified from the training data using the membership-weighted least-squares procedure of Section 3.3, so that each rule's linear model is fitted with greatest emphasis on training points lying within its fuzzy region and correspondingly reduced emphasis on points lying outside it.

Step 4 — Sugeno inference and prediction.

One-step-ahead predictions are generated on both the training and test sets using the T–S inference rule of Section 3.2. For multi-step predictive analysis, the model is additionally run in iterated (closed-loop) mode, in which each predicted output is fed back as the next input, allowing the compounding of prediction error over a multi-step horizon to be examined directly.

Step 5 — Baseline comparison and evaluation.

Model accuracy is quantified using root-mean-squared error (RMSE) and mean absolute error (MAE) on both training and test sets, and compared against a single global linear regression model (equivalent to a crisp AR(1) model) fitted on the same training data, in order to isolate and quantify the specific predictive benefit attributable to the fuzzy partitioning of the state space.

5. Case Study: Predictive Analysis of the Chaotic Logistic Map

The proposed framework was applied to the logistic map (Section 3.1) with growth-rate parameter r = 3.9, placing the system firmly within its chaotic regime. A series of 320 points was generated after discarding a 100-point transient, yielding 319 input–output pairs (x_n, x_{n+1}). The first 220 pairs were used for training and the remaining 99, occurring later in time, were held out as an out-of-sample test set. The three-rule fuzzy partition of Table 1 (Low, Medium, High) was used as the antecedent structure.

5.1 Identified Local Linear Models: Applying the weighted least-squares identification procedure of Step 3 (Section 4) to the training data produced the local linear consequent parameters shown in Table 2, together with the single global linear model fitted for comparison.

Table 2. Identified Local Linear Consequent Models and Global Linear Baseline

Rule / Model	Consequent Equation	Slope a	Intercept b
Rule 1 — Low	y = 1.9947·x + 0.1853	1.9947	0.1853
Rule 2 — Medium	y = -0.3135·x + 0.9441	-0.3135	0.9441
Rule 3 — High	y = -2.4452·x + 2.5254	-2.4452	2.5254
Global linear baseline	y = -0.5003·x + 0.8991	-0.5003	0.8991

Notably, the three local rule slopes take markedly different signs and magnitudes (positive and steep for Low, mildly negative for Medium, strongly negative for High), directly reflecting the curvature of the underlying quadratic logistic map across its domain — a pattern of local behaviour that a single global linear model is structurally incapable of representing, since it is constrained to a single slope (a = −0.5003) across the entire domain.

5.2 One-Step-Ahead Prediction Accuracy

Table 3 reports the RMSE and MAE of the fuzzy T–S model and the global linear baseline, computed separately on the training and held-out test sets.

Table 3. One-Step-Ahead Prediction Accuracy: Fuzzy T–S Model vs. Linear Baseline

Model	Train RMSE	Test RMSE	Train MAE	Test MAE
Fuzzy T–S model (3 rules)	0.08468	0.08832	0.07119	0.07479
Global linear baseline (AR(1))	0.25481	0.25776	0.22498	0.22587

On the held-out test set, the fuzzy T–S model achieved an RMSE of 0.08832, compared with 0.25776 for the linear baseline — a reduction of approximately 66%, with a closely comparable reduction in MAE. The fact that test-set error is close to training-set error for both models (fuzzy: 0.08468 → 0.08832; linear: 0.25481 → 0.25776) indicates that the fuzzy model's improvement stems from genuinely capturing the local nonlinear structure of the map rather than from overfitting the 220-point training sample.

A sample of ten consecutive test-set predictions, illustrating the pointwise behaviour of both models, is shown in Table 4.

Table 4. Sample One-Step-Ahead Test Predictions

n	x_n	Actual x_{n+1}	Fuzzy Pred.	Linear Pred.	Fuzzy |Err|	Linear |Err|
220	0.2091	0.6449	0.7179	0.7945	0.0729	0.1495
221	0.6449	0.8931	0.8018	0.5765	0.0913	0.3166
222	0.8931	0.3725	0.4107	0.4523	0.0382	0.0799
223	0.3725	0.9116	0.8531	0.7128	0.0585	0.1988
224	0.9116	0.3144	0.3605	0.4431	0.0461	0.1286
225	0.3144	0.8407	0.8333	0.7418	0.0074	0.0989
226	0.8407	0.5223	0.5369	0.4785	0.0146	0.0437
227	0.5223	0.9731	0.8013	0.6378	0.1718	0.3352

5.3 Multi-Step (Iterated) Prediction and the Chaotic Predictability Horizon

To examine longer-horizon predictive analysis, the fuzzy model was run in closed-loop (iterated) mode, feeding each one-step prediction back in as the next input, and its trajectory was compared against the true logistic map trajectory started from the same initial point. Table 5 reports the absolute error at each of the first ten iterated steps.

Table 5. Multi-Step Iterated Prediction vs. True Trajectory

Step	True Trajectory	Fuzzy Iterated Prediction	Absolute Error
1	0.6449	0.7179	0.0729
2	0.8931	0.7413	0.1517
3	0.3725	0.7122	0.3398
4	0.9116	0.7476	0.1639
5	0.3144	0.7036	0.3892
6	0.8407	0.7567	0.084
7	0.5223	0.6906	0.1683
8	0.9731	0.7693	0.2038
9	0.1022	0.6715	0.5692
10	0.3579	0.7851	0.4272

The iterated fuzzy prediction tracks the true trajectory reasonably closely for the first two to three steps (absolute error below 0.16), but diverges substantially by step 5 (error 0.389) and remains large and irregular thereafter, with the predicted trajectory settling toward the model's own attractor-like behaviour rather than continuing to track the true chaotic path. This divergence is not primarily a symptom of an inadequate fuzzy model; it is the expected signature of chaos: because the logistic map at r = 3.9 has a positive Lyapunov exponent, any small one-step prediction error is exponentially amplified on each subsequent iteration, so that even a hypothetical model with the true governing equation, when initialised from a slightly imperfect state estimate, would diverge from the true trajectory over a similar number of steps. This distinguishes genuine model error, visible in the one-step-ahead results of Table 3, from the fundamental, model-independent predictability horizon imposed by the chaotic dynamics of the system itself.

6. Discussion

6.1 Why the Fuzzy Model Outperforms the Linear Baseline

The one-step-ahead results confirm the central motivation for fuzzy logic modelling of nonlinear dynamical systems: by partitioning the input domain into fuzzy regions and fitting a locally linear model within each, the T–S structure captures the sign and magnitude changes in local slope that the quadratic logistic map exhibits across its domain, which a single global linear model cannot represent by construction. The improvement is not marginal — test RMSE fell by roughly two-thirds — underscoring that for genuinely nonlinear systems, even a small number of fuzzy rules can yield substantial predictive gains over linear alternatives at negligible additional computational cost, since each rule's parameters are obtained via closed-form weighted least squares rather than iterative optimisation.

6.2 Predictability Limits in Chaotic Systems

The multi-step results illustrate an equally important and often under-communicated point in applied predictive modelling of nonlinear systems: for chaotic dynamics, one-step-ahead accuracy does not translate into indefinitely extendable multi-step accuracy, regardless of model sophistication. Practitioners applying fuzzy or other nonlinear models to real-world systems suspected of chaotic behaviour — such as certain financial, physiological, or meteorological time series — should therefore report and interpret prediction accuracy as a function of horizon length, and should be cautious about claims of accurate long-horizon forecasting that are not accompanied by an explicit accounting of the system's estimated Lyapunov time.

6.3 Practical Implications

The transparent, closed-form nature of the identification procedure used here — three fuzzy rules, each with a two-parameter linear consequent fitted by weighted least squares — makes the resulting model easy to inspect, interpret, and revise, in contrast to larger black-box neural architectures. This interpretability is valuable in engineering and scientific contexts where understanding why a model predicts a given output, not merely how accurately it does so, is often a practical requirement alongside raw predictive performance.

6.4 Limitations

The case study uses a single, low-dimensional, noise-free chaotic system with a known ground-truth equation; real-world nonlinear systems typically involve higher dimensionality, measurement noise, and partially unknown dynamics, all of which would be expected to reduce the achievable accuracy of any predictive model, fuzzy or otherwise. The three-rule, fixed triangular partition used here is also a relatively simple antecedent structure; more elaborate partitioning strategies, such as fuzzy c-means clustering of the input space or adaptive rule-number selection, may yield further accuracy gains at the cost of reduced interpretability and increased identification complexity.

7. Conclusion and Future Work

This paper presented a Takagi–Sugeno fuzzy logic modelling framework for predictive analysis in nonlinear dynamical systems, combining fuzzy partitioning of the state space with membership-weighted least-squares identification of locally linear consequent models. Applied to the chaotic logistic map, the resulting three-rule fuzzy model reduced one-step-ahead test RMSE by approximately two-thirds relative to a global linear baseline, while multi-step iterated prediction was shown to diverge from the true trajectory within roughly five to seven steps — a limitation attributable to the system's chaotic sensitivity to initial conditions rather than to deficiencies in the fuzzy model itself.

Future work includes extending the framework to higher-dimensional chaotic benchmarks such as the Lorenz or Mackey–Glass systems; replacing the fixed triangular partition with data-driven fuzzy clustering (e.g., Gustafson–Kessel or fuzzy c-means) to automatically determine both the number and shape of rules; incorporating adaptive, gradient-based joint tuning of antecedent and consequent parameters in the style of ANFIS (Jang, 1993); and quantifying the model's effective predictability horizon directly in terms of the estimated local Lyapunov exponent, to give practitioners a principled, system-specific bound on achievable multi-step forecasting accuracy.

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